

Stick-slip dynamics of a granular layer under shear

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Stick-slip regime of shear granular flows is studied theoretically and numerically. Numerical experiments are carried out for a thin Couette cell using soft-particle molecular dynamics code in two dimensions. We apply order parameter theory of partially fluidized granular flows and find a good agreement with simulations and experiments by Nasuno *et al.*

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Densely packed granular assemblies under shear undergo a first-order phase transition: from static they become fluidized and the flow ensues. At smaller values of the shear stress, the granular “fluid” freezes, and the flow terminates. These melting-freezing transitions play a key role in granular friction. Nasuno *et al.* [1] studied the motion of a heavy plate pushed above a thin granular layer via a soft spring with a constant velocity of the pulling point. At large pulling speed, the plate moves with a constant velocity, whereas at smaller speeds, the motion of the plate is intermittent: long periods of *sticks* are followed by short *slip* events. Similar stick-slip oscillations have been observed in other instances of granular friction, for example when an object is moving through granular medium [2]. The mechanism of the stick-slip behavior of the granular layer was first addressed by Hayakawa [3] who introduced a global order parameter (OP) which characterized the phase state of the layer. In this model (as in the earlier model of friction [4]) the hysteretic OP dynamics is controlled by the plate speed [5]. The model yielded a stick-slip behavior qualitatively similar to Ref. [1], however it failed to describe the transition to the stationary sliding motion at larger pulling speeds. In Ref. [6] we applied our phenomenological theory to the same problem. Unlike Ref. [3], our OP is local, and it is controlled by the Ginzburg-Landau (GL) free energy depending on the local ratio of shear to normal stress in the spirit of the Mohr-Coulomb yield criterion. Our model described fluidization transition in a thin near-surface layer leading to stick-slip oscillations at small pulling speeds, and stationary flow at larger speeds. However, in Ref. [6] we used phenomenological equation for the OP introduced *ad hoc*, so the agreement with the experiment [1] was only qualitative.

In this paper we apply our recent continuum theory of partially fluidized flows [7] to the quantitative description of granular friction. We also take into account the plate inertia. In agreement with experiment [1], we find a discontinuous transition from the stick slip to the stationary motion and reproduce the friction force-plate velocity phase loop characterizing slip events. Furthermore, we describe the structure of the granular flow underneath the plate during slip. Our theory is compared with two-dimensional (2D) soft-particle molecular dynamics simulations.

We consider a granular layer in a planar Couette geometry driven by a moving upper plate of mass M and length L_x . We use a fixed coordinate frame with horizontal axis x , ver-

tical axis y , and the origin at the top of the layer. We assume no slip between the plate and the granular layer, so the plate velocity V coincides with the granular velocity $V(0)$ at $y=0$. The plate is pulled with a constant speed V_0 via a soft linear spring with elasticity constant K , so the force acting on the plate from the spring is $F=K(x-V_0t-x_0)$, where x is the position of the plate front and x_0 is its initial position (at $t=0$ the spring is unloaded). All variables and parameters are scaled by the gravity acceleration g , particle size d_p , and mass m_p . The equations of motion for the plate read

$$\dot{x}=V, \quad m\dot{V}=\sigma-\kappa(x-V_0t-x_0), \quad (1)$$

where $m=M/L_x$, $\kappa=K/L_x$, and σ is the (negative) yx shear stress component (we assume that it does not depend on y , which is a reasonable approximation based on our numerical simulations). For a given σ , the OP ρ is found from the GL equation [7]

$$\epsilon\partial_t\rho=D\partial_y^2\rho-(\rho-1)(\rho^2-2\rho_*\rho+\rho_*^2e^{-A(\delta^2-\delta_*^2)}) \quad (2)$$

with $\rho_*=0.6$, $\delta_*=0.25$, $A=25$, $D=2$, $\epsilon=0.02$. The control parameter δ is the ratio of shear to normal stress, $\delta(y)=-\sigma/p(y)=-\sigma/(m-y)$. We assume no-flux boundary conditions $\rho'(0)=\rho'(-L_y)=0$. The characteristic feature of this equation is the bistability in the certain range of δ , $0.25<\delta<0.3$, where the last term has three roots $\rho_1<\rho_2<\rho_3=1$ among which ρ_1 corresponding to the fluid phase and $\rho_3=1$ corresponding to a solid phase are stable, and ρ_2 is unstable. At $\delta<0.25$ the only stationary uniform solution corresponds to a solid phase $\rho=\rho_3=1$, and at $\delta>0.3$ it only exists in a fluidized phase $\rho=\rho_2$, however within $0.25<\delta<0.3$ both phases may coexist. This feature in fact gives rise to the stick-slip oscillations of the plate. To close the system, we integrate the constitutive relation [7]

$$(1-\rho)^q\sigma=-\mu_f V'(y) \quad (3)$$

with no-slip boundary condition $V(-L_y)=0$, so

$$\sigma=-\mu_f V\left[\int_{-L_y}^0(1-\rho)^q dy\right]^{-1}. \quad (4)$$

Here $\mu_f\approx 12$ is the constant viscosity coefficient of the fluid subsystem and $q\approx 2.5$ is the scaling constant found from 2D

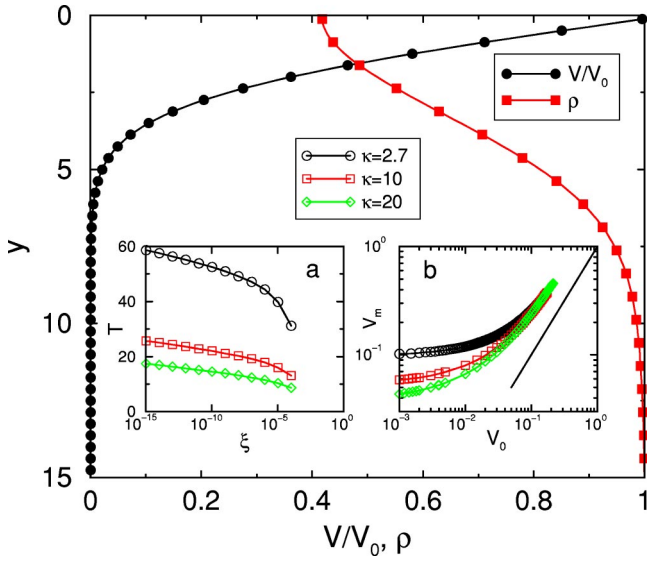


FIG. 1. Vertical profiles of the order parameter ρ and horizontal velocity v normalized by the plate velocity V for $m=20$, $\kappa=2.7$, $V_0=0.2$. *Inset a*: stick-slip oscillations period as a function of the noise magnitude ξ subtracted from the order parameter at every time step for $m=20$, $V_0=0.2$ and three values of κ . *Inset b*: maximum plate velocity during slip V_m vs V_0 for three values of spring constant. Solid line corresponds to the continuous sliding ($V_m=v_0$).

simulations of the thin Couette flow [7]. Equation (4) can be used only within the slip event, because during the stick phase the OP $\rho \rightarrow 1$, and velocity $V(0) \rightarrow 0$, so Eq. (4) becomes indeterminate. In this case, the inertia of the plate can be ignored, the shear stress coincides with the spring force,

$$\sigma = -\kappa(x - V_0 t - x_0), \quad (5)$$

and V can be found from the constitutive relation (3),

$$V = V(0) = -\mu_f^{-1} \sigma \int_{-L_y}^0 (1 - \rho)^q dy. \quad (6)$$

Equations (1)–(6) were integrated numerically using the finite difference method. We switch from Eqs. (2), (5), and (6) to Eqs. (1), (2), and (4) when the plate velocity reaches a small threshold value $V_{tr} = 10^{-3}$ (the dynamics of the system is not sensitive to the specific value of V_{tr}). In a long stick phase the plate velocity turns into machine zero and the order parameter “freezes” in the unstable fixed point $\rho = \rho_3 = 1$ from which it cannot escape without perturbations. To avoid this spurious behavior, we subtracted a small random value [uniformly distributed between 0 and $\xi = O(10^{-6})$] from the order parameter at every time step. This correction accounts for small rearrangements occurring in the granular system under a nonstationary load. It remedies the problem, however the magnitude of the perturbation affects (albeit only logarithmically) the period between the slips and, correspondingly, the magnitude of the spring deflection. Inset *b* in Fig. 1 shows the dependence of the oscillations period on the noise term magnitude ξ [8].

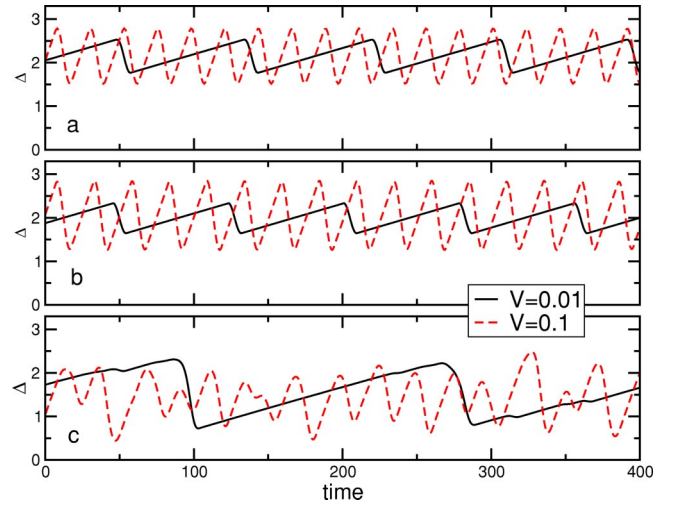


FIG. 2. Relaxation oscillations of the deflection Δ in GL theory (a), reduced equation (9) (b), and MD simulations (c) for $m=20$, $\kappa=2.7$, $\xi=10^{-6}$: relaxation oscillations for $V_0=0.01$, quasiperiodic oscillations at $V_0=0.1$.

We chose the parameters of our model (mass per unit length, m , and the spring constant per unit length κ) based on experimental values [1]. A direct comparison between our 2D system and the 3D experiment is difficult, however we can assume that our 2D system represents a one-particle diameter “slice” of the full 3D system. In this case, the mass and the spring constant in the experimental system should be scaled by the area of the top plate. Using nondimensional units based on the gravity acceleration, particle mass and diameter, we find that mass of the plate 10 g corresponds to $m \approx 20$, and the spring constant $k=135$ N/m corresponds to $\kappa \approx 2.7$. These values of m and κ have been used in most of our calculations. We also ran simulations for stiffer springs with $\kappa=10$ and 20. The velocities are scaled by $(gd_p)^{1/2}$. Thus for smooth particles explored in Ref. [1] the velocity scale is ≈ 30 mm/s.

The main control parameter of this model is the pulling velocity V_0 . At large V_0 we obtain a stationary near-surface shear flow. The vertical profiles of velocity and order parameter for $V_0=0.2$ are shown in Fig. 1 (corresponding dimensional velocity is 6.2 mm/s). The velocity profile is almost linear near the surface, and below the shear layer there is an exponentially decaying tail of the creep flow. The order parameter is small within the shear layer and approaches 1 at large depth.

At small velocities $V_0 \rightarrow 0$ the model exhibits relaxation oscillations reminiscent of the dry friction between two solids. The spring deflection $\Delta = \sigma/\kappa$ grows almost linearly with no flow until it reaches a certain threshold value after which the near-surface layer fluidizes, and the ensuing shear flow relieves the accumulated stress. After that the layer freezes again, and the process repeats [Fig. 2(a)]. In agreement with experiments, in the inertia dominated regime at larger pulling speeds, the deflection of the spring becomes almost sinusoidal [see Fig. 2(b)]. Inset *b* in Fig. 1 indicates that the transition from sliding to stick slips is discontinuous, and there is a range of velocities V_0 in which sliding and stick slips coexist.

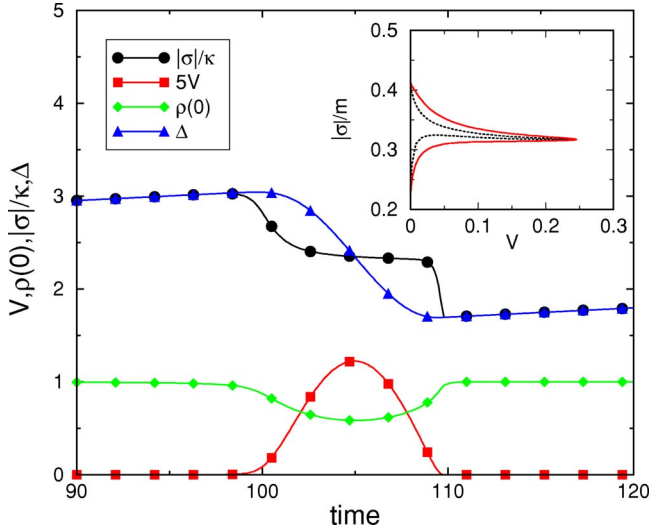


FIG. 3. Structure of an individual slip event $m=20$, $K=2.7$, $V_0=0.001$: plate velocity V , near-surface order parameter ρ , shear stress $|\sigma|$, and the spring force F . Inset: normalized shear stress $|\sigma|/M$ vs plate velocity V for two values of ϵ : $\epsilon=0.02$ (solid line), $\epsilon=0.1$ (dotted line).

Figure 3 depicts the structure of an individual slip. As the normalized shear stress below the plate reaches the critical value $\delta(0) \approx 0.32$, the order parameter starts to deviate from 1, and the plate begins to move. The most interesting feature of the slip event is a nonmonotone decay of the shear stress σ . During the initial “melting” the order parameter decreases, therefore the integral in Eq. (4) starts to grow. Meanwhile, the plate velocity still remains small due to inertia. Therefore, at this stage the shear stress first quickly drops. Later the stress fall stops because the order parameter saturates at a certain low level, and the plate velocity reaches a maximum. At the final phase of the slip, the velocity decays while the order parameter is still below 1 so the shear stress drops rapidly again, until it reaches the lower critical value. Thereafter, the order parameter quickly returns to 1 and the velocity to zero, the granular layer freezes and the new stick phase ensues. Inset in Fig. 3 shows the phase trajectory σ vs V which is remarkably similar to the experimental loop reported by Nasuno *et al.* [1]. The main quantitative difference is that the critical value of the normalized shear stress in the experiment was 0.65, whereas our theory calibrated by two-dimensional molecular dynamics (MD) simulations gives 0.32. This difference can possibly be explained by the fact that experiments were done in 3D geometry. It is interesting to see that the continuum theory describes the “precursor” of a slip: small deviation of the order parameter from 1 begins long before the actual slip occurs (see Fig. 4). The detailed structure of the slip event depends rather sensitively on the parameter ϵ which characterizes the relaxation time of the order parameter. Inset in Fig. 3 shows the friction stress—plate velocity diagram for two different values of ϵ .

Equations (1)–(6) can be reduced to the set of equations for the velocity and position of the plate V, x and the thickness of fluidized layer ξ . Equation (2) can be written as

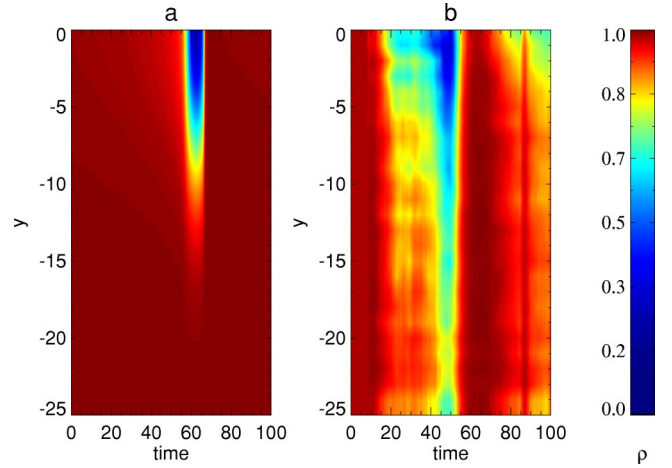


FIG. 4. Space-time plots of ρ for $m=20$, $\kappa=2.7$, $V_0=0.01$: continuum model (a), numerical simulations (b).

$$\epsilon \partial_t \rho = D \partial_y^2 \rho - (\rho - 1)(\rho - \rho_1)(\rho - \rho_2) - (\rho - 1)(\rho_*^2 e^{-A(\delta^2 - \delta_*^2)} - \rho_1 \rho_2), \quad (7)$$

where $\rho_1 + \rho_2 = 2\rho_*$. If the roots $\rho_{1,2}$ obey the condition $1 - \rho_2 = \rho_2 - \rho_1$ yielding $\rho_2 = (2\rho_* + 1)/3, \rho_1 = (4\rho_* - 1)/3$, Eq. (7) without the last term (that will be treated as a perturbation) possesses a stationary front solution

$$\rho = \frac{1 - \rho_1}{2} \left(1 - \tanh \left[\frac{(y + \xi)(1 - \rho_1)}{\sqrt{8D}} \right] \right) + \rho_1. \quad (8)$$

Here $\xi(t) > 0$ is the position of the fluidization front or thickness of the fluidized layer. Due to the dependence of δ on y the variable ξ becomes a function of time. The evolution of ξ can be obtained from standard orthogonality conditions ($m \gg 1$ is assumed)

$$\dot{\xi} = C_0 \sigma^2 \left(1 - \frac{2\xi}{m} \right) - C_1, \quad (9)$$

where

$$C_0 = \frac{3\rho_*^2 A \sqrt{2D}}{\epsilon m^2 (1 - \rho_1)}, \quad C_1 = \frac{[(1 - \rho_*)^2 + 9\rho_*^2 A \delta_*^2] \sqrt{2D}}{3\epsilon (1 - \rho_1)}.$$

For the parameters of Eq. (2) one has $C_0 = 12.65$, $C_1 = 326.4$. From Eq. (4) we find $\sigma \approx -\mu_f V [(1 - \rho_1)\xi]^{-q}$. For $\xi = O(1)$ the front solution loses its validity and correspondingly, Eq. (9) needs to be corrected. Formally, for $\xi < \xi_c \ll 1$ Eq. (9) can be replaced by the equation for most unstable linear perturbations $\dot{\xi} = \lambda \xi$, where $\lambda \approx [\rho_*^2 A (\sigma^2/m^2 - \delta_*^2) - (1 - \rho_*)^2]/\epsilon$ is the growth rate of small perturbations following from Eq. (2). Then the corresponding solution can be matched with the front solution valid for large ξ . The structure of the resulting solution does not depend on the cutoff value for $\xi_c \rightarrow 0$. Numerically solved Eqs. (9) and (1) are in good quantitative agreement with continuum theory, Eqs. (1)–(6), see Fig. 2, however they are significantly simpler

and allow for analytical treatment. For instance, they can be used to find the duration of slip event. During the slip event one can neglect $\dot{\xi}$ in Eq. (9), and the shear stress is almost constant, $\sigma \approx \sigma_{slip} \approx \sqrt{C_1/C_0}$. Then the slip duration T_{slip} can be found from two equations (1) assuming $\sigma = \sigma_{slip}$, with initial condition $V(0)=0$ and $x(0)=x_{stick}$, where x_{stick} is defined by the end of the stick phase, $x_{stick} = \sigma_{max}/\kappa$, σ_{max} is the value of stress at the end of stick. Solving linear Eqs. (1) one finds after simple algebra ($\omega = \sqrt{\kappa/m}$ is the oscillation frequency)

$$T_{slip} = \frac{2}{\omega} \arctan \left[\frac{\omega(\sigma_{max} - \sigma_{slip})}{V_0 \kappa} \right]. \quad (10)$$

For $V_0=0.01$ one obtains $T_{slip} \approx 8$, which is in a good agreement with numerical solution. For $V_0 \rightarrow 0$ the slip time approaches $T_{slip} \rightarrow \pi/\omega$ and is determined by the response of the mass-spring system.

Our continuum theory, Eqs. (1)–(6), has been calibrated in Ref. [7] using 2D soft-particle molecular dynamics simulations of a stationary thin Couette flow between two plates without gravity. We compared it with simulations of a stationary near-surface boundary layer in a thick Couette cell under gravity. It is interesting to compare the *non stationary* stick-slip dynamics as described by this theory with MD simulations. We performed 2D simulations of a granular layer driven by a heavy upper plate through a soft spring analogous to the experiment [1]. Similar calculations have been done in Refs. [9,10] using soft-particle MD with only sliding friction taken into consideration. We believe for the adequate description of the stick-slip dynamics it is important to take into account dry friction between the grains. To this end, we used the well-established approach [11,12] based on the Cundall-Strack model [13]. The details of our numerical algorithm and parameters are given in Ref. [7].

We simulated up to 3000 polydisperse grains in a rectangular box of length $L_x=100$ with periodic boundary conditions in x direction and horizontal boundaries made of particles with the same material properties but twice larger than an average grain. Figure 2(c) shows the time series of the

spring deflection for two pulling speeds. They are quite similar to experimental results in that they have a form of well separated slip events at small pulling speeds and quasisinusoidal oscillations near the onset of sliding motion at large speeds. The main difference with the theory and the experiment is that the slip events are rather irregular. This can be explained by the fact that in our small system the plate was held by only a few (30–40) grains in a stuck position. Besides, there is a quantitative difference in the characteristic time between slips which is caused by the fact that in numerical simulations the deflection during the slip falls to a lower value than theoretically expected value $mg\delta_*/\kappa$. We believe that this effect is caused by the dilation of the thin near-surface layer of grains which permits the plate to slip farther than according to our incompressible theory. Nevertheless, individual slip events bear strong qualitative and quantitative similarity to the slip events found in continuum model as well as in experiments [1]. Precursor to the slip which was found in Ref. [1] can be clearly seen in the evolution of the OP averaged over horizontal coordinate during the slip event (Fig. 4).

In summary, we applied the continuum theory of partially fluidized granular flows developed earlier on the basis of 2D molecular dynamics simulations of stationary granular shear flows to an essentially nonstationary problem of granular friction in the regime of stick-slip oscillations. Our theory to date is only calibrated using the 2D molecular dynamics simulations, and that precludes a more rigorous comparison of the theory with experiments. However, on a qualitative level, the theory exhibits a remarkable agreement with experiment [1] both in the bifurcation structure (it shows a subcritical transition from stick-slip behavior to constant sliding motion with increase of the pulling speed), and in the detailed structure of the slip event as indicated by Fig. 3.

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- merge and disappear at $\delta > 0.3$ through a saddle-node bifurcation. Since in this case there is no fixed point at large δ , the granular layer fluidizes and the plate slips rapidly after the control parameter δ exceeds the critical bifurcation value. We ran simulations with such a modified order parameter equation without noise (data not shown) and found a qualitatively similar stick-slip behavior without noise excitation.
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